ASSESSING THE IMPACT OF ECCENTRICITY AND SPINS FOR GRAVITATIONAL WAVE ASTROPHYSICS

Antoine Klein

Institut d'Astrophysique de Paris, 98 bis Bd Arago, 75014 Paris, France

ABSTRACT

We present a family of frequency-domain gravitational waveforms for precessing binaries on eccentric orbits valid for generic spin configurations and magnitudes. These waveforms are fast to generate and provide excellent agreement with time-domain waveforms computed via a discrete Fourier transform. These provide a realistic solution for the search for generic precessing binaries in gravitational wave data analysis, due to their low computational cost. These can also be used to assess under which circumstances circular waveforms can be used in gravitational wave parameter estimation.

In general, compact object binary systems with arbitrary spin configurations undergo spin-orbit precession, which affects their emission of gravitational waves in a non-trivial manner. Furthermore, the presence of a non-zero eccentricity has a large impact on the waveform, both on the amplitudes of the different harmonics present, but also on the phase evolution of the system. The equations of motion are such that four different timescales arise in the problem: the orbital timescale $T_{\rm orb} \sim \omega^{-1}$, the spin-precession timescale $T_{\rm sp} \sim \omega^{-5/3}$, the periastron-precession timescale $T_{\rm pp} \sim \omega^{-5/3}$ and the radiation reaction timescale $T_{\rm r.r.} \sim \omega^{-8/3}$, where ω is the binary orbital frequency.

The orbit of an eccentric binary system can be described in General Relativity using a quasi-Keplerian parametrization:

$$r = a(1 - e_r \cos u) + f_r(v), \tag{1}$$

$$\phi = (1+k)v + f_\phi(v), \tag{2}$$

$$\tan\frac{v}{2} = \sqrt{\frac{1+e_\phi}{1-e_\phi}}\tan\frac{u}{2}, \tag{3}$$

$$l = u - e_t \sin u + f_t(v), \tag{4}$$

$$\dot{l} = n + f_n(v), \tag{5}$$

where (r, ϕ) is a polar representation of the separation vector in the orbital plane, a is the semimajor axis, u is the eccentric anomaly, v is the true anomaly, l is the mean anomaly, n is the mean motion, e_r, e_ϕ, and e_t are eccentricity parameters, and the functions $f_i(v)$ are general relativistic corrections.

We can express the phases ϕ and l as linearly growing and periodic components, $\phi = \lambda + W_\phi$ and $l = \bar{l} + W_l$. To do so, we express

$$\dot\lambda = (1+k)n, \tag{6}$$

$$\dot{\bar{l}} = n, \tag{7}$$

$$W_l = \int f_n(v)dt, \tag{8}$$

$$W_\phi = (1+k)(v - l + W_l) + f_\phi(v). \tag{9}$$

With the addition of the evolution equations for the mean orbital frequency $\omega = \dot\lambda$, the eccentricity $e = e_t$, the spins and the orbital angular momentum, we can express the waveform in the time domain. It is most conveniently expressed as a series of harmonics of the orbital phase, with amplitudes dependent on the eccentric anomaly:

$$h(t) = F_+(t)h_+(t) + F_\times(t)h_\times(t), \tag{10}$$

$$h_{+,\times}(t) = \sum_{n\in\mathbb{Z}} H^{(n)}_{+,\times}(\omega, e, e\cos u, e\sin u)e^{in(\phi+\phi_T)}, \tag{11}$$

where $F_{+,\times}$ are the antenna pattern functions and their time dependence originate from the evolution of the orbital angular momentum, and ϕ_T is an additional phase due to precession.

In order to transform this signal into terms suitable for an analytic approximation of its Fourier transform, we have to invert the quasi-Kepler equation (4). We can do it as

$$u = l + \sum_{s\geq 1} A_s \sin sl, \tag{12}$$

provided that we expand the amplitudes A_s for small eccentricities in addition to its post-Newtonian expansion.

Armed with this, we can now expand the time-domain signal into harmonics of the linearly growing phases as

$$h_{+,\times}(t) = \sum_{n\in\mathbb{Z}}\sum_{p\in\mathbb{Z}} H^{(p,n)}_{+,\times} e^{i(n\lambda + pl)}. \tag{13}$$

where the amplitudes $H_{+,\times}$ are varying on the spin-precession timescale, and are expressed as a series in the orbital frequency and the eccentricity.

We can then separate the orbital timescale and the periastron-precession timescale by expressing the mean motion as a function of the orbital frequency:

$$l = \lambda - \delta\lambda, \tag{14}$$

$$\delta\dot\lambda = \frac{k}{1+k}\omega. \tag{15}$$

Finally, we can express the time-domain waveform into a series of harmonics of the mean orbital phase, with an amplitude varying on the spin-precession and periastron-precession timescales.

$$h_{+,\times}(t) = \sum_{n\in\mathbb{Z}} H^{(n)}_{+,\times} e^{in\lambda}, \tag{16}$$

$$H^{(n)}_{+,\times} = \sum_{m\in\mathbb{Z}} G^{(m,n)}_{+,\times} e^{im\delta\lambda} e^{i(m+n)\phi_T}. \tag{17}$$

We can then use a SUA transformation to approximate the Fourier transform of the signal in the presence of spin precession:

$$\begin{aligned}\tilde{h}_{+,\times}(f) &= \int h_{+,\times}(t)e^{2\pi i f t}dt \\ &= \sum_{n\geq 1}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \sqrt{\frac{2\pi}{n\ddot\lambda}} a_k G^{(m,n)}_{+,\times}(t_{n,m} + kT_n) \\ &\quad \times e^{i[2\pi f t_{n,m} - n\lambda(t_{n,m}) - m\delta\lambda(t_{n,m}) - \pi/4]},\end{aligned} \tag{18}$$

$$2\pi f = n\dot\lambda(t_{n,m}) + m\delta\dot\lambda(t_{n,m}), \tag{19}$$

$$T_n = \left(n\ddot\lambda\right)^{-1/2}. \tag{20}$$

We can then Taylor expand the definition of $t_{n,m}$ and the Fourier phase around $t_{n,0}$ to get the final form of the Fourier transform:

$$\tilde{h}_{+,\times}(f) = \sum_{n\geq 1}\sqrt{\frac{2\pi}{n\ddot\lambda}} G^{(n)}_{+,\times} e^{i[2\pi f t_n - n\lambda(t_n) - \pi/4]}, \tag{21}$$

$$2\pi f = n\dot\lambda(t_n), \tag{22}$$

$$\begin{aligned}G^{(n)}_{+,\times} &= \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_k G^{(m,n)}_{+,\times}(t_n + \Delta t_{n,m} + kT_n) \\ &\quad \times e^{i\Delta\Psi_{n,m}}.\end{aligned} \tag{23}$$

This result can be used to compute reliably and efficiently the Fourier transform of a gravitational wave signal for the inspiral phase, given any solution for the orbital phase, the eccentricity and the orbital angular momentum as a function of time. We find that this waveform is very reliable to approximate the Fourier transform of the gravitational-wave signal in the presence of precession, up to an initial eccentricity of 0.03 if we only keep the amplitude at leading order in the eccentricity, and up to an initial eccentricity of 0.3 if we keep the amplitudes at the level of $\mathcal{O}(e^6)$, as shown in Fig. 1.

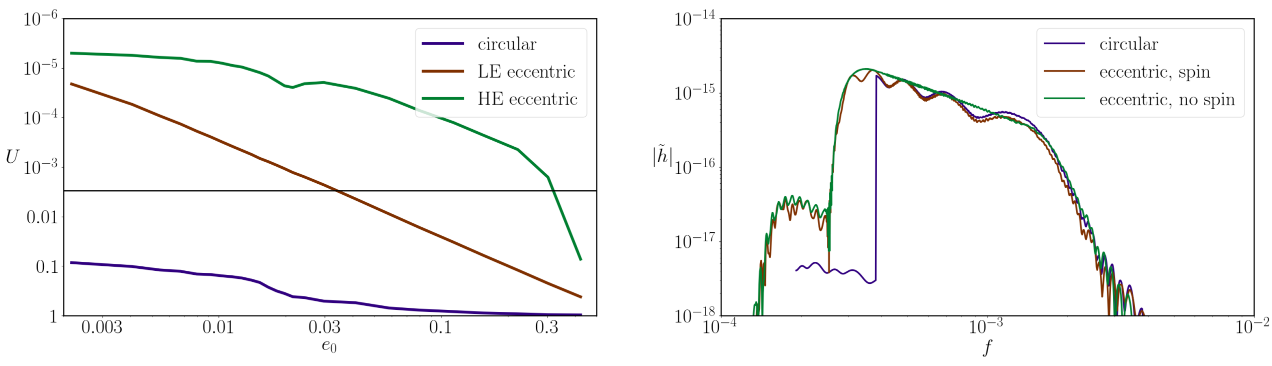

Figure 1: Left: distributions of the unfaithfulness $U = 1 - F$ for a circular model, an eccentric model with amplitudes at the lowest order in the eccentricity (LE), and an eccentric model with amplitudes at high order in the eccentricity ($\mathcal{O}(e^6)$, HE), all compared with a time-domain waveform computed via a discrete Fourier transform. The faithfulness F is a measure of the similarity between two signals, takes values between -1 and 1, and a faithfulness of 1 indicates identical signals for the same parameters. Right: comparison between the amplitude of three waveforms, one circular, one eccentric with spins, and one eccentric without spins, otherwise with the same parameters. The initial eccentricity of the eccentric waveforms here is 0.03. We can see that neglecting the spins leads to the loss of crucial modulations of the waveform that can be used to extract extra information from it, while neglecting the eccentricity leads to a mischaracterization of the frequency evolution, likely to inflict large biases on the recovered parameters.